

THE PREQUANTUM LINE BUNDLE ON THE MODULI SPACE OF FLAT $SU(N)$ CONNECTIONS ON A RIEMANN SURFACE AND THE HOMOTOPY OF THE LARGE N LIMIT.

LISA C. JEFFREY, DANIEL A. RAMRAS, AND JONATHAN WEITSMAN

ABSTRACT. We show that the prequantum line bundle on the moduli space of flat $SU(2)$ connections on a closed Riemann surface of positive genus has degree 1. It then follows from work of Lawton and the second author that the classifying map for this line bundle induces a homotopy equivalence between the stable moduli space of flat $SU(n)$ connections, in the limit as n tends to infinity, and $\mathbb{C}P^\infty$. Applications to the stable moduli space of flat unitary connections are also discussed.

1. INTRODUCTION

Let G be a simply connected compact Lie group and let Σ be a closed oriented 2-manifold of genus $g > 0$. In [8] Ramadas, Singer and Weitsman construct a line bundle \mathcal{L} over the moduli space of gauge equivalence classes of flat connections $\mathcal{A}_F(\Sigma)/\mathcal{G}$ on a trivial G -bundle on Σ . This bundle possesses a natural connection, whose curvature is a scalar multiple of Goldman's symplectic form.

The purpose of this paper is to compute the degree (that is, the first Chern class) of the line bundle described in [8] in the case $G = SU(2)$. Our main theorem is

Theorem 1.1. *The degree of the line bundle is 1.*

As we will explain, the second integral cohomology group of the moduli space is infinite cyclic, and the Theorem implies that the first Chern class of \mathcal{L} is a generator. There is in fact a preferred generator (depending on the orientation of Σ), which agrees with $c_1(\mathcal{L})$.

In view of Question 5.6 of [6], Theorem 1.1 has the following corollary:

Corollary 1.2. *Let Σ be a closed oriented 2-manifold of genus $g > 0$. Let \mathcal{G}_n be the gauge group of the trivial $SU(n)$ -bundle on Σ , and let $\mathcal{A}_F^{SU(n)}(\Sigma)$ denote the space of flat connections on this bundle. The classifying maps for the line bundles $\mathcal{L} \rightarrow \mathcal{A}_F^{SU(n)}(\Sigma)/\mathcal{G}_n$ induce a homotopy equivalence $\text{colim}_{n \rightarrow \infty} \mathcal{A}_F^{SU(n)}(\Sigma)/\mathcal{G}_n \simeq \mathbb{C}P^\infty$.*

It was previously shown in [6] that the stable moduli space $\text{colim}_{n \rightarrow \infty} \mathcal{A}_F^{SU(n)}(\Sigma)/\mathcal{G}_n \cong \text{colim}_{n \rightarrow \infty} \text{Hom}(\pi_1 \Sigma, SU(n))/SU(n)$ is a $K(\mathbb{Z}, 2)$ space, and hence is homotopy equivalent to $\mathbb{C}P^\infty$. This corollary gives a geometric viewpoint on this homotopy equivalence. In Section 4, we also obtain a geometric viewpoint on the homotopy equivalence $\text{colim}_{n \rightarrow \infty} \text{Hom}(\pi_1 \Sigma, U(n))/U(n) \simeq (S^1)^{2g} \times \mathbb{C}P^\infty$ from [9].

LJ was partially supported by a grant from NSERC.

DR was partially supported by a grant from the Simons Foundation (#279007).

JW was partially supported by NSF grant DMS-12/11819.

Another approach to the study of these moduli spaces in the $n \rightarrow \infty$ limit was recently proposed by Hitchin [4]. It would be interesting to understand the relation between these approaches.

Our computation of $c_1(\mathcal{L})$ in genus 1 (Section 3) is similar to Kirk–Klassen [5, Theorem 2.1].¹ We also note that Drezet–Narasimhan [3] determined a generator for the Picard group of $\mathcal{A}_F^{SU(n)}(\Sigma)/\mathcal{G}_n$, viewed as the moduli space of semistable holomorphic bundles of rank n on Σ with trivial determinant. However, we do not know whether the line bundle considered in this paper agrees with the bundle considered in [3] and hence it is not clear how to deduce our results from [3]. The topological methods in this article avoid delicate questions about algebraic singularities in the moduli space.

Acknowledgements: The second author thanks Simon Donaldson for suggesting that the results of [9] should be connected to Goldman’s symplectic form. Additionally, we thank Jacques Hurtubise for asking about the higher genus case considered in Section 5.

2. THE CHERN-SIMONS LINE BUNDLE

The line bundle from [8] is defined using the Chern-Simons cocycle ([8], p. 411) $\Theta : \mathcal{A} \times \mathcal{G} \rightarrow \mathbb{C}$ defined by

$$\Theta(A, g) = \exp i(CS(\mathbf{A}^g) - CS(\mathbf{A})).$$

Here G is a simply connected, compact Lie group with Lie algebra \mathfrak{g} , $\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g})$ is the space of connections on the trivial G -bundle over Σ , and $\mathcal{G} = C^\infty(\Sigma, G)$ is the gauge group of this bundle. The Chern-Simons functional $CS(\mathbf{A})$ is defined by

$$CS(\mathbf{A}) = \frac{1}{4\pi} \int_N \text{Trace}(\mathbf{A}d\mathbf{A} + \frac{2}{3}\mathbf{A}^3)$$

where N is a 3-manifold with boundary Σ and $g \in \mathcal{G} = C^\infty(\Sigma, G)$. We have chosen extensions \mathbf{A} and \mathbf{g} of A and g (respectively) over the bounding 3-manifold N (the existence of \mathbf{g} relies on simple connectivity of G). It is shown in [8] that the Chern-Simons cocycle $\Theta(A, g)$ is independent of the choice of these extensions. We define a line bundle \mathcal{L} over $\mathcal{A}_F/\mathcal{G}$ as a \mathcal{G} -equivariant bundle over the space of flat connections \mathcal{A}_F , where $g \in \mathcal{G}$ acts on $\mathcal{A} \times \mathbb{C}$ by

$$g : (A, z) \mapsto (A^g, \Theta(A, g)z).$$

The definition of \mathcal{L} is

$$\mathcal{L} = \mathcal{A}_F \times_{\mathcal{G}} \mathbb{C}.$$

The symplectic form $\hat{\Omega}$ on \mathcal{A} is defined by (see [8], p. 412):

$$(1) \quad \hat{\Omega}(a, b) = \frac{i}{2\pi} \int_{\Sigma} \text{Trace}(a \wedge b)$$

for $a, b \in \Omega^1(\Sigma, \mathfrak{g})$. Notice that on the affine space \mathcal{A} , the symplectic form is a constant quadratic form; it does not depend on choosing a point in \mathcal{A} .

¹Kirk and Klassen conclude that $c_1(L) = -1$. The discrepancy can be explained using the footnote regarding signs in Section 3 of the present article.

3. DEGREE OF THE CHERN-SIMONS LINE BUNDLE IN GENUS 1

Let N be a three-manifold with boundary Σ .

The symplectic form on \mathcal{A} from (1) descends to a symplectic form Ω on $\mathcal{A}_F^s/\mathcal{G}$, where \mathcal{A}_F denotes the flat connections and \mathcal{A}_F^s the irreducible flat connections.

The authors of [8] exhibit a unitary connection $\hat{\omega}$ on the prequantum line bundle over \mathcal{A}_F :

$$(2) \quad \hat{\omega}(a) = \frac{i}{4\pi} \int_{\Sigma} \text{Trace}(A \wedge a)$$

whose curvature is $\hat{\Omega}$. This is done on p. 412 of [8]. The proof uses the fact that the derivative of the Chern-Simons function is

$$dCS_A(v) = \frac{1}{4\pi} \left(\int_N 2\text{Trace}(v \wedge F_A) - \int_{\Sigma} \text{Trace}(A \wedge v) \right)$$

for $v \in T_A\mathcal{A} = \mathcal{A} = \Omega^1(N, \mathfrak{g})$. This follows from a straightforward calculation using Stokes' theorem. The above expression restricts on \mathcal{A}_F to

$$dCS_A(v) = -\frac{1}{4\pi} \int_{\Sigma} \text{Trace}(A \wedge v) = i\hat{\omega}(v)$$

(recalling (2)). It is shown on p. 412 of [8] (second paragraph) that $\hat{\omega}$ is the pullback of a connection ω on $\mathcal{A}_F \times_{\mathcal{G}} \mathbb{C}$. This is demonstrated by introducing a vertical vector field Y for the action of \mathcal{G} , and showing that

$$i_Y \hat{\omega} = L_Y \hat{\omega} = 0$$

so $\hat{\omega}$ is basic, and therefore descends to a 1-form on $\mathcal{A}_F/\mathcal{G}$.

For the rest of the section, we restrict to $G = SU(2)$. Let x, y be the flat coordinates on the genus 1 surface (see the proof of Lemma 3.1 for more details). Inside the space \mathcal{A} we can consider the space \mathcal{W} of all connections of the form $adx + bdy$ where $a, b \in \text{Lie}(T)$ and T is the maximal torus of $SU(2)$.

Now \mathcal{W} is a subspace of \mathcal{A} so the bundle \mathcal{L} restricts to \mathcal{W} as a bundle with connection. This bundle is invariant under that part of the gauge group that preserves \mathcal{W} . This consists of $(\mathbb{Z} \times \mathbb{Z}) \ltimes \mathbb{Z}_2$. Here $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ is identified with the gauge transformation $(e^{ix}, e^{iy}) \mapsto e^{imx} e^{iny}$, and $\mathbb{Z}_2 = \{\pm 1\}$ is the Weyl group of $SU(2)$.

Taking the quotient by $\mathbb{Z} \times \mathbb{Z}$ we get a bundle \mathcal{L}' on $T \times T = \mathcal{W}/(\mathbb{Z} \times \mathbb{Z})$ with a connection ω' . We will show, via a direct computation (Lemma 3.1), that the curvature Ω' of this connection has integral equal to $-4\pi i$, and using Chern-Weil theory we will be able to conclude that \mathcal{L}' has degree 2, while \mathcal{L} has degree 1.

The computation goes as follows.

Lemma 3.1. *We have*

$$\int_{T \times T} \Omega' = -4\pi i.$$

Proof. Let $X = \text{diag}(i, -i) \in \mathfrak{su}(2)$. Then $\text{Trace}(X^2) = -2$.

Let Γ be a fundamental domain for the action of $\mathbb{Z} \times \mathbb{Z}$ on $\text{Lie}(T) \oplus \text{Lie}(T)$. Parametrize Γ by $(x, y) \in [0, 2\pi] \times [0, 2\pi]$. Here x corresponds to $\exp(xX)$ (where $\exp : \text{Lie}(T) \rightarrow T$ is the exponential map). The vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ on Γ are identified with the constant vector field X on T so with the above formula (1) for $\hat{\Omega}$ we have that

$$\begin{aligned} \Omega'_{(x,y)} &:= \Omega'_{(x,y)}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \Omega_{(x,y)}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{i}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \text{Trace}(X^2) dx dy \\ &= -4\pi i. \end{aligned}$$

The value of $\Omega'_{(x,y)}$ is independent of x and y , so the integral of Ω' over $T \times T$ (integrating using an area form of total area 1) is also $-4\pi i$. \square

Proof of Theorem 1.1 in genus 1. By Lemma 3.1, the cohomology class $[\Omega'] \in H^2(T \times T; \mathbb{R})$ associated to Ω' is $-4\pi i \alpha$, where $\alpha = \frac{1}{(2\pi)^2} [dx \wedge dy] \in H^2(T \times T; \mathbb{R})$ denotes the fundamental class. By Chern–Weil theory², we have

$$[\Omega'] = -2\pi i c_1(\mathcal{L}'),$$

so $c_1(\mathcal{L}') = \frac{2}{(2\pi)^2} [dx \wedge dy] = 2\alpha$, and \mathcal{L}' has degree two.

The generator of $W = \mathbb{Z}/2\mathbb{Z}$ acts on $T \times T$ by complex conjugation on each factor, inducing a quotient map $f: T \times T \rightarrow (T \times T)/W \cong S^2$. Our bundle \mathcal{L}' is \mathbb{Z}_2 -equivariant, and it descends to the bundle \mathcal{L} on $(T \times T)/W$. So $f^*(\mathcal{L}) = \mathcal{L}'$, and hence $\deg(\mathcal{L}) = \frac{1}{\deg(f)} \deg(\mathcal{L}')$. An elementary calculation (e.g. using a \mathbb{Z}_2 -equivariant CW complex structure on $T \times T$) shows that $\deg(f) = 2$, completing the proof. \square

The key point is that we have computed the degree on $T \times T$, which has a canonical smooth manifold structure, so we can use Chern–Weil theory. The proof of Theorem 1.1 in higher genus will be given in Section 5.

4. THE CONJECTURE OF LAWTON AND RAMRAS ON THE CHERN-SIMONS LINE BUNDLE

Let Σ be a closed oriented 2-manifold of genus $g > 0$. Let $\mathcal{G}_{SU(n)} = \mathcal{G}_{SU(n)}(\Sigma)$ and $\mathcal{G}_{U(n)} = \mathcal{G}_{U(n)}(\Sigma)$ denote the gauge groups of the trivial $SU(n)$ and $U(n)$ -bundles on Σ (respectively), and let $\mathcal{A}_F^{SU(n)}(\Sigma)$ and $\mathcal{A}_F^{U(n)}(\Sigma)$ denote the spaces of flat $SU(n)$ - and $U(n)$ -connections on these bundles. Define

$$\mathcal{M}_U(\Sigma) = \text{colim}_n \mathcal{A}_F^{U(n)}(\Sigma) / \mathcal{G}_{U(n)} \text{ and } \mathcal{M}_{SU}(\Sigma) = \text{colim}_n \mathcal{A}_F^{SU(n)}(\Sigma) / \mathcal{G}_{SU(n)}.$$

We refer to these as the *stable moduli spaces* of flat unitary (or special unitary) connections over Σ . Let $\mathcal{L}_n \rightarrow \mathcal{A}_F^{SU(n)}(\Sigma) / \mathcal{G}_{SU(n)}$ denote the prequantum line bundle. As n varies, these bundles are compatible with the inclusions $SU(n) \hookrightarrow SU(n+1)$, and

²In [7, Appendix C], the Chern–Weil formula for characteristic classes is stated without signs, because they use a version of Fubini’s Theorem with signs [7, p. 304]. Since we have integrated using the usual version of Fubini’s Theorem, we need a sign in our formula for $c_1(\mathcal{L}')$.

hence induce a line bundle $\mathcal{L}_\infty \rightarrow \mathcal{M}_{SU}(\Sigma)$, which we call the stable prequantum line bundle.

The homotopy types of the stable moduli spaces were determined in [6, 9]:

$$\mathcal{M}_U(\Sigma) \simeq (S^1)^{2g} \times \mathbb{C}P^\infty, \quad \mathcal{M}_{SU}(\Sigma) \simeq \mathbb{C}P^\infty.$$

These results are computational, and rely on the uniqueness of Eilenberg–MacLane spaces; that is, no explicit homotopy equivalences between these spaces have been constructed. Here we offer bundle-theoretic descriptions of these homotopy equivalences.

Remark 4.1. *The proof that $\mathcal{M}_U(\Sigma)$ and $\mathcal{M}_{SU}(\Sigma)$ have the homotopy types stated above relies on an independent result showing that these spaces have the homotopy types of CW complexes. In the unitary case, this is proven in [9, Lemma 5.7], and the same argument works in the special unitary case.*

Theorem 4.2. *The classifying map*

$$\mathcal{M}_{SU}(\Sigma) \longrightarrow \mathbb{C}P^\infty$$

for the stable prequantum line bundle \mathcal{L}_∞ is a homotopy equivalence.

Proof. Writing $\Sigma = \Sigma' \# T$, where T is a torus, the quotient map $\Sigma \rightarrow \Sigma/\Sigma' \cong T$ and the inclusions $SU(2) \hookrightarrow SU(n)$ together induce a map

$$\mathcal{A}_F^{SU(2)}(T)/\mathcal{G}_{SU(2)}(T) \longrightarrow \mathcal{M}_{SU}(\Sigma),$$

which induces an isomorphism on $H^2(-; \mathbb{Z})$ by [6, Theorem 5.3]. We have shown in the previous section that the classifying map for the bundle $\mathcal{L}_2 \rightarrow \mathcal{A}_F^{SU(2)}(T)/\mathcal{G}_{SU(2)}(T)$ induces an isomorphism on $H^2(-; \mathbb{Z})$. Since the classifying map for \mathcal{L}_∞ restricts to a classifying map for \mathcal{L}_2 , we find that the classifying map for \mathcal{L}_∞ must also induce an isomorphism on $H^2(-; \mathbb{Z})$. But up to homotopy, this is a self-map of $\mathbb{C}P^\infty$, and any self-map of $\mathbb{C}P^\infty$ that induces an isomorphism on $H^2(-; \mathbb{Z})$ is a homotopy equivalence. \square

We now turn to the unitary case.

Corollary 4.3. *There exists a line bundle $\mathcal{L}^U \rightarrow \mathcal{M}_U(\Sigma)$ that restricts to $\mathcal{L}_\infty \rightarrow \mathcal{M}_{SU}(\Sigma)$, and if α is a classifying map for \mathcal{L}^U , then the map*

$$\mathcal{M}_U(\Sigma) \xrightarrow{(\det, \alpha)} (S^1)^{2g} \times \mathbb{C}P^\infty$$

is a homotopy equivalence, where $\det: \mathcal{M}_U(\Sigma) \rightarrow (S^1)^{2g}$ is the determinant map.

Proof. Let $f: \mathcal{M}_{SU} \rightarrow \mathbb{C}P^\infty$ be a classifying map for \mathcal{L}_∞ (so that by Theorem 4.2, f is a homotopy equivalence). We simply need to prove the existence of a homotopy commutative diagram

$$\begin{array}{ccc} \mathbb{C}P^\infty \simeq \mathcal{M}_{SU} & \xrightarrow{i} & \mathcal{M}_U \simeq \mathbb{C}P^\infty \times (S^1)^{2g} \\ & \searrow f \quad \swarrow \alpha & \\ & \mathbb{C}P^\infty & \end{array}$$

in which i is the natural inclusion and α induces an isomorphism on π_2 . Then we can set \mathcal{L}^U to be the pullback, under α , of the universal bundle over $\mathbb{C}P^\infty$. To see

that (\det, α) is a homotopy equivalence, note that \det is split by the inclusion of $(S^1)^{2g} \cong \text{Hom}(\pi_1(\Sigma), U(1))/U(1)$ into $\mathcal{M}_U(\Sigma)$. Therefore, on fundamental groups \det_* is a surjection between free abelian groups of rank $2g$, and we conclude that \det_* is an isomorphism. It follows that (\det, α) is a weak equivalence. Since $\mathcal{M}_U(\Sigma)$ has the homotopy type of a CW complex (Remark 4.1), the Whitehead Theorem implies that (\det, α) is in fact a homotopy equivalence.

Now we construct the desired map α . It is shown in [6, Theorem 5.3] that i induces an isomorphism on second homotopy groups. We claim that for any map $i: \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty \times (S^1)^{2g}$ that induces an isomorphism on π_2 , and any homotopy equivalence $f: \mathbb{C}P^\infty \xrightarrow{\sim} \mathbb{C}P^\infty$, there exists a map $\alpha: \mathbb{C}P^\infty \times (S^1)^{2g} \rightarrow \mathbb{C}P^\infty$ such that α induces an isomorphism on π_2 and the diagram

$$(3) \quad \begin{array}{ccc} \mathbb{C}P^\infty & \xrightarrow{i=(i_1, i_2)} & \mathbb{C}P^\infty \times (S^1)^{2g} \\ & \searrow \simeq & \swarrow \alpha \\ & \mathbb{C}P^\infty & \end{array}$$

commutes up to homotopy. Since i induces an isomorphism on π_2 , so does i_1 . Hence i_1 is a homotopy equivalence $\mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$. Setting $\alpha = f \circ i_1^{-1} \circ p_1$, where $p_1: \mathbb{C}P^\infty \times (S^1)^{2g} \rightarrow \mathbb{C}P^\infty$ is the projection and i_1^{-1} is a homotopy inverse to i_1 , one checks that α has the desired properties. \square

5. THE DEGREE OF THE LINE BUNDLE IN HIGHER GENUS

Let \mathcal{L}_g denote the prequantum line bundle on the moduli space $\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}$ of flat connections on a trivial $SU(2)$ -bundle over the genus g surface Σ^g . We now show that \mathcal{L}_g has degree 1 for every genus g surface ($g > 0$), not just $g = 1$ (thereby completing the proof of Theorem 1.1). This statement is meaningful, since we have:

Lemma 5.1. *For any $g \geq 1$, we have $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. In [6], it was proven that $\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}$ is simply connected (and a second proof of this fact was given in [1]). Now, triviality of $H_1(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$ implies that $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$ is torsion-free (by the Universal Coefficient Theorem), and a simple direct analysis of the Poincaré polynomial of $\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}$, as determined by Cappell–Lee–Miller [2], shows that $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$ has rank 1. \square

A map $f: \Sigma^g \rightarrow \Sigma^h$ induces a map

$$f^\#: \mathcal{A}_F^{SU(2)}(\Sigma^h)/\mathcal{G} \rightarrow \mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G},$$

and as noted in [8, Remark 3, p. 412], if f has degree 1 then $(f^\#)^*(\mathcal{L}_g) = \mathcal{L}_h$. This implies that

$$(f^\#)^*(c_1(\mathcal{L}_g)) = c_1(\mathcal{L}_h),$$

and taking $h = 1$ we find that $(f^\#)^*(c_1(\mathcal{L}_g)) = c_1(\mathcal{L}_1) = 1$ (by the result in Section 3). Now Lemma 5.1 implies that $c_1(\mathcal{L}_g)$ is a generator of $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$. This completes the proof of Theorem 1.1. \square

Remark 5.2. *The results in [6] in fact show that the map*

$$H^2(\mathcal{M}_{SU}(\Sigma^g); \mathbb{Z}) \longrightarrow H^2(\mathcal{M}_{SU}(\Sigma^1); \mathbb{Z})$$

is determined by $f^: H^2(\Sigma^1) \rightarrow H^2(\Sigma^g)$. Thus a choice of generator in $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^1)/\mathcal{G}; \mathbb{Z})$ and a choice of orientations on Σ^1 and Σ^g give a choice of generator in $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$, and the above discussion shows that this generator coincides with $c_1(\mathcal{L}_g)$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA
E-mail address: jeffrey@math.toronto.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, IUPUI, INDIANAPOLIS, IN 46202, USA
E-mail address: dramras@math.iupui.edu

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115, USA
E-mail address: j.weitsman@neu.edu